Comment/Commentaire

Comment on: Curling rock dynamics — The motion of a curling rock: inertial vs. noninertial reference frames

Mark R.A. Shegelski and Matthew Reid

Abstract: We examine the approach used and the results presented in a recent publication (Can. J. Phys. **76**, 295 (1998)) in which (*i*) a noninertial reference frame is used to examine the motion of a curling rock, and (*ii*) the lateral motion of a curling rock is attributed to left–right asymmetry in the force acting on the rock. We point out the important differences between describing the motion in an inertial frame as opposed to a noninertial frame. We show that a force exhibiting left–right asymmetry in an inertial frame *cannot* explain the lateral motion of a curling rock. We also examine, as was apparently done in the recent publication, an *effective* force that has left–right asymmetry *in a noninertial, rotating frame*. We show that such a force is *not* left–right asymmetric *in an inertial frame*, and that any lateral motion of a curling rock attributed to the *effective* force in the noninertial frame is actually due to a *real* force, in an inertial frame, which has a net nonzero component transverse to the velocity of the center of mass. We inquire as to the physical basis for the transverse component of this *real* force. We also examine the motion of a rotating cylinder sliding over a smooth surface for which there is no melting: we show that the motion is easily analyzed in an inertial frame and that there is little to be gained by considering a rotating frame. We relate the results for this simple case to the more involved problem of the motion of a curling rock: we find that the motion of curling rocks is best studied in *inertial* frames. Perhaps most importantly, we show that the approach taken and the results presented in the recent publication lead to *predicted motions* of curling rocks that are *in disagreement with observed motions* of real curling rocks.

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Résumé : Nous examinons l'approche utilisée et les résultats obtenus dans une récente publication (Can. J. Phys. **76**, 295 (1998)) dans laquelle (*i*) un référentiel non-inertiel est utilisé pour étudier le mouvement d'une pierre de curling et (*ii*) la dérive latérale de la pierre est attribuée à une asymétrie gauche-droite dans la force agissant sur la pierre. Nous soulignons les différences importantes dans la description du mouvement dans un référentiel inertiel et dans

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M.R.A. Shegelski[1] **and M. Reid.** Department of Physics, University of Northern British Columbia, 3333 University Way, Prince George, BC V2N 4Z9, Canada.

[1] Corresponding author: **Telephone:** (250) 960–6663; **FAX:** (250) 960–5545; **e-mail:** mras@unbc.ca

un référentiel non-inertiel. Nous montrons qu'une force montrant une asymétrie gauche–droite dans un référentiel inertiel ne peut pas expliquer la dérive latérale de la pierre. Nous examinons aussi, ce qui avait apparamment été fait dans la précédente publication, une force efficace avec une asymétrie gauche-droite dans un référentiel non-inertiel en rotation. Nous montrons qu'une telle force n'a pas d'asymétrie gauche–droite dans un référentiel inertiel et que tout déplacement latéral de la pierre attribué à la force efficace dans le référentiel non-inertiel est en fait dû à une vraie force dans le référentiel inertiel avec une composante non-nulle transverse à la direction de la vitesse du centre de masse. Nous nous interrogeons sur les causes physiques de la composante transverse d'une telle force. Nous examinons également le mouvement d'un cylindre en rotation sur une surface lisse sans fonte superficielle : nous montrons que le mouvement est facilement analysable dans un référentiel inertiel et qu'il n'y a rien à gagner à l'étudier dans un référentiel non-inertiel en rotation. Nous relions l'étude de ce cas simple à l'étude du mouvement d'une pierre de curling : nous trouvons qu'il est préférable d'étudier le mouvement de la pierre dans un référentiel inertiel. Plus important peut-être, nous montrons que l'approche utilisée et les résultats présentés dans la précédente publication mènent à des prédictions en désaccord avec le mouvement observé de la pierre.
[Traduit par la rédaction]

1. Introduction

In the sport of curling, cylindrical granite rocks slide over pebbled ice. Only a brief account of the aspects of curling most relevant to this paper will be conveyed here. The reader may consult any of numerous books on curling to more fully understand this intriguing sport. Information may also be obtained from, for example, the Canadian Curling Association [1].

Two chief aspects relevant to the discussion in this paper are the shape of the bottom of the rock, and the nature of "pebbled" ice. The rocks have a small contact area with the ice: the bottom of the rock is curved and hollowed out, so that only a thin annulus (of diameter 12.5 cm and width 3 to 5 mm) makes contact with the ice. The ice surface consists of many rounded protrusions, with adjacent valleys, and is called pebbled ice. The consequence is that kinetic melting of the ice results as the rock moves over it, and a thin liquid film is nested between (portions of) the contact annulus and the ice surface [2].

The motion exhibited by curling rocks is quite interesting, in several respects. Perhaps most interesting is the trajectory of the rocks: slowly rotating rocks moving over the sheet of ice do not move in a straight line; instead, the path is curved (hence the name "curling"). Moreover, the direction in which the rocks curl is *opposite* to the direction of lateral displacement of other rotating cylinders sliding over solid surfaces. For example, an overturned cylindrical drinking glass, projected over a smooth surface and rotating counterclockwise (as viewed from above and behind), will move laterally *to the right*, whereas a curling rock, sliding over pebbled ice, and rotating counterclockwise, will curl *to the left*.

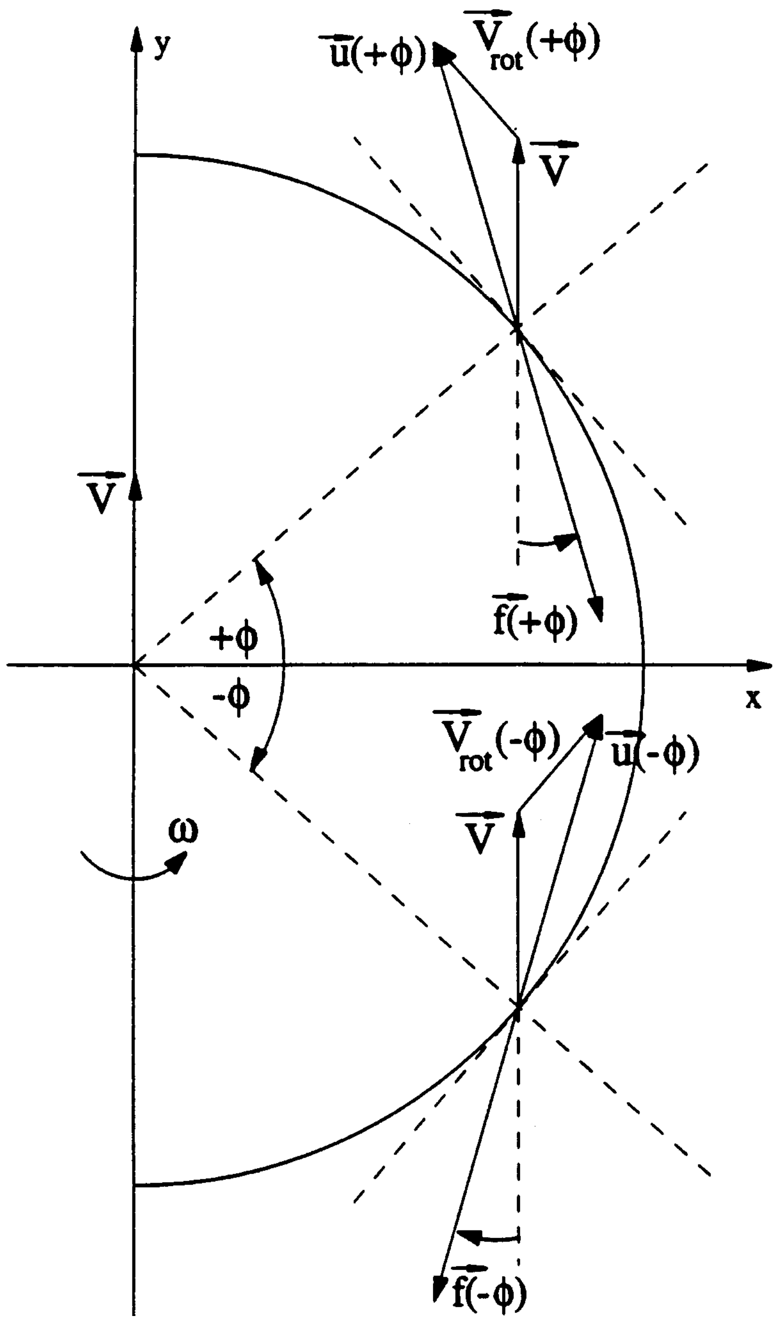
Perhaps the most interesting question to ask is: why does a curling rock curl, and why does it curl in the direction that it does?

Two distinct models [2,3] have successfully accounted for the lateral motion of curling rocks by using front–back asymmetry. Another publication [4] claims the lateral motion can be accounted for instead by a left–right asymmetry. By "left–right" and "front–back" asymmetry we mean the following. The instantaneous velocity of the center of mass of the rock may be used to break the rock up into halves: left and right halves, or front and back. Left–right asymmetry means the forces acting around the contact annulus are symmetric front and back but asymmetric left and right (see Fig. 1). Similarly, front–back asymmetry means the forces are asymmetric front and back.

Five main points will be made in this paper.

1. We show that the lateral motion of slowly rotating curling rocks cannot be explained by a force that has left–right asymmetry in an inertial frame; consequently, the lateral motion is due to front–back asymmetry. We show that left–right asymmetry, in an inertial frame, gives no net force lateral to the direction of motion of the rock, and thus results in straight–line motion.

Fig. 1. Forces acting on portions of the right half of a curling rock for the case of left–right asymmetry. The instantaneous center of mass velocity $\boldsymbol{v}$ is in the $+y$-direction, and the rock is rotating counterclockwise as viewed from above. Shown are the velocities relative to the ice of two portions of the contact annulus, located at angles $\pm\phi$. The net velocities are simply $\boldsymbol{u}(\pm\phi) = \boldsymbol{v} + \boldsymbol{v}_{\text{rot}}(\pm\phi)$, where $|\boldsymbol{v}_{\text{rot}}(\pm\phi)| = r\omega$, r is the radius of the contact annulus, and the directions of $\boldsymbol{v}_{\text{rot}}(\pm\phi)$ are tangential to the contact annulus, as shown. The two forces, $\boldsymbol{f}(\pm\phi)$, in directions opposite to $\boldsymbol{u}(\pm\phi)$, are equal in magnitude and make the same angle with the y-axis, but in opposite senses. The two forces combine to give a net force in the $-y$-direction, and *no net force transverse to* $\boldsymbol{v}$.



2. Early in this paper, we will assume that the results in ref. 4 are correct, and we will inquire as to the consequences of those results. In particular, we will use the expressions presented in ref. 4 to derive the equation for the real force, $\boldsymbol{F}$, in the inertial frame, that is required to obtain the description proposed in ref. 4. We will find that the expression for the net nonzero component, $\boldsymbol{F}_{\mathrm{T}}$, transverse to the velocity of the center of mass of the rock, has a rather unorthodox form, and we will point out that no physical derivation of this force $\boldsymbol{F}_{\mathrm{T}}$ has as yet been given. We will inquire as to the physical basis for $\boldsymbol{F}_{\mathrm{T}}$: we will see that the form of $\boldsymbol{F}_{\mathrm{T}}$ is physically unreasonable, and leads to predicted motions that are in clear disagreement with observed motions.

3. We also consider an *effective* force having left–right asymmetry in a rotating, noninertial reference frame. We do so because it seems that this may be what was done in ref. 4. If such an effective force is to give lateral motion, as observed in the inertial frame, then the lateral motion is due to a *real* force, in the inertial frame, that has a net nonzero component, $\boldsymbol{F}_{\mathrm{T}}$, transverse to the velocity of the center of mass of the rock. We show that the real force in the inertial frame is not left–right asymmetric; consequently, the lateral motion of the rock is actually due to front–back asymmetry, just as in the case of refs. 2 and 3. We again find that the motions predicted by ref. 4 are in disagreement with observed motions.

4. We carefully consider the motion of a rotating cylinder sliding over a smooth, flat surface for which there is no melting. We show that the motion is readily analyzed in an inertial frame, and that there is little to be gained by examining the motion in a rotating frame. We also show that, whereas all aspects of the motion are easily addressed by working in an inertial frame, problems can easily arise if one attempts to determine all aspects of the motion by working exclusively in a rotating frame. Since such problems can arise in a case where the forces and the motion in an inertial frame are unequivocal, we suggest that it is best to work in an inertial frame in the considerably more complicated case of the motion of a curling rock.

5. Perhaps *the most important point of this paper* is the following. The approach taken in ref. 4, and the results presented in ref. 4, lead to *predicted motions* of curling rocks that are *in disagreement with observed motions* of real curling rocks.

Our two principal conclusions will be (*i*) that the approach and the results in ref. 4 are incorrect, and (*ii*) that working in an inertial frame is much more appropriate than attempting to use a noninertial frame.

2. Left–right asymmetry

We summarize, and comment briefly on, what was done in ref. 4.

(1.) The following was presented as a left–right asymmetric force acting on the contact annulus of a slowly rotating curling rock:

$$\boldsymbol{f}(\phi) = -\mu(\phi) M g \boldsymbol{e}(\phi) \tag{1}$$

with

$$\mu(\phi) = \mu(1 - b\cos\phi) \tag{2}$$

where μ and b are positive constants, and $\boldsymbol{e}(\phi)$ is the instantaneous direction of motion, relative to the underlying ice surface, of the portion of the contact annulus located at angle ϕ; the angle ϕ is measured counterclockwise from the direction perpendicular to the instantaneous velocity of the center of mass, $\boldsymbol{v}$, with $0 \le \phi \le 2\pi$, and $\phi = \pi/2$ is the direction of $\boldsymbol{v}$. (See Fig. 1; note that $\boldsymbol{e}(\phi) \equiv \boldsymbol{u}(\phi)/|\boldsymbol{u}(\phi)|$.)

(2.) The reasoning given in ref. 4 for (2) is as follows. The motion of the contact annulus of the rock over the ice causes kinetic melting of the ice, resulting in a thin liquid film. The liquid film results in less friction. (The idea of the thin liquid film was discussed previously in both refs. 2 and 3, especially so in ref. 2.) The most melting will occur at the point moving fastest relative to the ice, i.e., where the "rotational" velocity (with magnitude $r\omega$) is added to $\boldsymbol{v}$ to give a net relative speed of $v + r\omega$ (this point is located at $\phi = 0$). The least melting will occur at the point having a net relative speed of $v - r\omega$ (i.e., at $\phi = \pi$). Equation (2) gives a coefficient of friction capturing this idea.

On reading this explanation, one is led to consider (1) and (2) to give the force acting on the rock *in an inertial frame*. However, it is not clear whether or not this was what was intended in ref. 4. Instead, it seems that the force given by (1) and (2), which has meaning *in an inertial frame*, may have been taken to be the effective force *in the rotating, noninertial frame*. We will discuss this more fully below.

(3.) In ref. 4, there is mention of two reference frames: an inertial frame, and a rotating, noninertial frame. No clear definition was given in ref. 4 for the frames used. From the results presented in ref. 4 one assumes the inertial frame to have its origin located at the point of release of the rock (i.e., at the initial position of the rock, where $v = v_0$ and $\omega = \omega_0$). It was stated in ref. 4 that the origins of the inertial and rotating frames coincide. From this, one might think that the rotating frame also has its origin at the release point of the rock, and that the rotating frame was to have an angular speed Ω relative to the inertial frame such that the rock would appear, in the rotating frame, to recede directly away from the origin, and to show no lateral motion, as viewed in the rotating frame. Another possible interpretation for the rotating frame, and a more likely one, is the following. At any instant, one is to use a frame that has its origin on the ice, beneath the center of mass of the rock, that rotates with an angular speed Ω relative to the inertial frame such that the rock exhibits no lateral motion in the rotating frame. Moreover, the origin of the rotating frame is stationary relative to the inertial frame; i.e., the origin of the rotating frame does not move with the rock. In this case, what one has is a continuum of rotating frames. We have looked at both possibilities, and have concluded that the latter of the two was most probably the one actually used in ref. 4, and by "the rotating frame" we will mean, unless otherwise stated, this second possibility.

Given that inertial and rotating frames were used in ref. 4, there are two possible interpretations of the force in (1): (*i*) that $\boldsymbol{f}(\phi)$ is the force in the inertial frame, and (*ii*) that $\boldsymbol{f}(\phi)$ is the "effective" force, in the noninertial reference frame, i.e., that the "force" in the equation

$$\boldsymbol{F}_{\text{eff}} = M\boldsymbol{a}_{\text{rot}}$$

with $\boldsymbol{a}_{\text{rot}}$ being the acceleration measured in the rotating frame, is obtained from (1) via

$$\boldsymbol{F}_{\text{eff}} = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\phi\, \boldsymbol{f}(\phi)$$

The reader will recall that the effective force in a noninertial frame consists of a combination of real forces, in an inertial frame, and *fictitious forces* in the noninertial frame that arise due to the acceleration and (or) rotation of the noninertial frame.

We will examine both possibilities (*i*) and (*ii*). We will find that (*i*) in the inertial frame, (1) and (2) give no net force transverse to the velocity $\boldsymbol{v}$ of the center of mass, and consequently that the rock moves in a straight line; (*ii*) in the rotating frame, if the rock is to move laterally (as viewed from the inertial frame), then the lateral motion is due to a *real* force that has a net nonzero component $\boldsymbol{F}_{\text{T}}$ transverse to $\boldsymbol{v}$ in the inertial frame. We will derive the expression for $\boldsymbol{F}_{\text{T}}$ that is required by the proposal made in ref. 4 and show that this expression for $\boldsymbol{F}_{\mathrm{T}}$ has not yet been given any physical justification. We also show that the form of $\boldsymbol{F}_{\mathrm{T}}$ is physically unreasonable, and predicts motions of curling rocks that are in disagreement with observed motions.

3. Left–right asymmetry in an inertial frame

It has already been reported in the literature [5] that a left–right asymmetry cannot give lateral displacement. That report was succinct, however, and to ensure the point is clear, we present here a more detailed analysis. To show definitively that (1) and (2) result in no net lateral force, consider Fig. 1. The figure shows the right half of the rock. The instantaneous center-of-mass velocity $\boldsymbol{v}$ is in the $+y$-direction, and the rock is rotating counterclockwise as viewed from above. The figure shows the velocities relative to the ice of two portions of the contact annulus, located at angles $\pm\phi$. The net velocities are simply $\boldsymbol{u}(\pm\phi) = \boldsymbol{v} + \boldsymbol{v}_{\mathrm{rot}}(\pm\phi)$, where $|\boldsymbol{v}_{\mathrm{rot}}(\pm\phi)| = r\omega$, and the directions of $\boldsymbol{v}_{\mathrm{rot}}(\pm\phi)$ are tangential to the contact annulus, as shown. The two forces, $\boldsymbol{f}(\pm\phi)$, also shown, are equal in magnitude and make the same angle with the y-axis, but in opposite senses. The two forces combine to give a net force in the $-y$-direction, and *no net force transverse to* $\boldsymbol{v}$. This simple exercise holds for all such pairs around the contact annulus, with the result that there is no net force lateral to the direction of motion of the center of mass. Consequently, a left–right asymmetric force in an inertial frame gives no lateral motion: the rock moves in a straight line.

Although it is clear, it must be stated: for a net external force $\boldsymbol{F}$ that is always in the direction opposite to the velocity $\boldsymbol{v}$ of the center of mass, the trajectory of the center of mass is a straight line.

4. Predicted vs. observed motions

In this section we will give the equations and results presented in ref. 4. We will examine the consequences of these equations and results. We will show that the approach taken and the results presented in ref. 4 lead to predicted motions of curling rocks that are in disagreement with the observed motions of actual curling rocks. In the following section of this paper, we will address the question of what was done in ref. 4 to lead to these incorrect results.

The following equation was used in ref. 4 to relate the accelerations of the center of mass of a curling rock in the inertial and rotating frames:

$$\boldsymbol{a} = \boldsymbol{a}' + \boldsymbol{\Omega} \times \boldsymbol{v} \tag{3}$$

$\boldsymbol{a}'$ was given by

$$\boldsymbol{a}' = \frac{1}{M}\frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\phi\, \boldsymbol{f}(\phi) \approx -\mu g \boldsymbol{e}_v \tag{4}$$

where $\boldsymbol{e}_v$ is a unit vector in the direction of $\boldsymbol{v}$. The following expression was given for Ω:

$$\Omega \approx b\frac{r}{R^2}(v_0 - v) \tag{5}$$

where

$$v(t) \approx v_0(1 - \mu g t/v_0) \tag{6}$$

with v_0 being the initial speed, r the radius of the contact annulus, and R the outer radius of the curling rock.

These equations were used in ref. 4 to calculate the acceleration $\boldsymbol{a}$ in the inertial frame. The trajectory of the rock in the inertial frame was given in ref. 4 by the following two equations:

$$|x(t)| = \frac{1}{2}b\frac{r v_0^2}{R^2 t_0}\left(\frac{t^3}{3} - \frac{t^4}{4t_0}\right) \tag{7}$$

and

$$y(t) = v_0 \left(t - \frac{t^2}{2t_0} \right) \tag{8}$$

where $t_0 = v_0/(\mu g)$.

One cannot tell, just by looking at (7) and (8), whether or not this is a physically reasonable result. However, by addressing the crucial physical question that arises, one can easily conclude that (7) and (8) are physically unacceptable.

The crucial question that arises is: *What is the real, physical force, $\boldsymbol{F}$, in the inertial frame, that gives rise to this motion*? One finds, by combining (3)–(5), that

$$\boldsymbol{F} = \boldsymbol{F}_v + \boldsymbol{F}_\mathrm{T} = -\mu M g \boldsymbol{e}_v - M b \frac{r}{R^2} (v_0 - v) v \boldsymbol{e}_\mathrm{T} \tag{9}$$

where $\boldsymbol{e}_\mathrm{T}$ is transverse to $\boldsymbol{e}_v$. The transverse component of this force is physically unreasonable; it is proportional to *the reduction in the speed of the rock since it was released*!

We will discuss in more detail this rather strange force, $\boldsymbol{F}$, that is required to give the trajectory specified by (7) and (8). Our main task in this section is to examine the consequences of the results presented in ref. 4.

A very serious consequence of these equations is the following physically unreasonable result. Two curling rocks, released from different initial positions, but such that they have the same angular and linear velocities at a given later time (i.e., are "side by side", rotating in the same sense, and moving with the same velocity), *will subsequently follow different paths*. To ensure clarity, we consider a specific example.

Consider the motions of two curling rocks released from different initial positions with different initial velocities, as follows. For example, one rock (rock A) is released from one end of the sheet of ice with an initial speed of $v_0^{(\mathrm{A})} = 2.5$ m/s. The speed of rock A decreases until, at time $t = t_1$ its speed is $v_0^{(\mathrm{A})}(t_1) \equiv v_1 = 0.75 v_0^{(\mathrm{A})} = 1.875$ m/s. At this time t_1, rock B is released with an initial speed $v_0^{(\mathrm{B})} = v_1 = 0.75 v_0^{(\mathrm{A})} = 1.875$ m/s, and in the same direction that rock A is traveling at time t_1; the release point of rock B is chosen such that the line connecting the centers of the two rocks is perpendicular to their velocities. In other words, the two rocks are, at time t_1, side by side, and are moving in the same direction, with the same speed v_1. Rock B is also released with an angular rotation in the same sense as that of rock A.

Note that, according to ref. 4, the trajectory of a rock does not depend on the *magnitude* of the angular velocity, only on its *direction*. Thus, it is not necessary, according to ref. 4, that rocks A and B have, at time t_1, the same angular speed; it is only required that they rotate in the same sense.

Using (7) and (8), one can easily calculate the x and y coordinates, in the inertial frame, of the final locations of the two rocks. Note that one must take into account that the motions of the two rocks are calculated with respect to their release points and their initial directions of motion.

The results are as follows. The final y coordinates of the two rocks are, to leading order, the same: $y_f^{(\mathrm{A})} \approx y_f^{(\mathrm{B})}$ (these are with respect to the release point of rock A). For example, taking $\mu = 0.0127$ and $g = 9.8$ m/s, $y_f^{(\mathrm{A})} \approx y_f^{(\mathrm{B})} \approx 25.1$ m.

However, the x coordinates are quite different. For example, taking $r = 0.0625$ m, $R = 0.14$ m, and $b = 0.003$, one finds $|x_f^{(\mathrm{A})}| \approx 1.0$ m whereas $|x_f^{(\mathrm{B})}| \approx 0.6$ m: a difference of about 0.4 m!

(These values of $|x_f^{(\mathrm{A})}|$ and $|x_f^{(\mathrm{B})}|$ are, again, with respect to the release point of rock A.)

Observations of motions of actual curling rocks, projected in the manner described above, are in clear disagreement with this prediction. One readily observes that $y_f^{(\mathrm{A})} \approx y_f^{(\mathrm{B})}$ *and also that* $x_f^{(\mathrm{A})} \approx x_f^{(\mathrm{B})}$! (In the example above, one readily observes that $|x_f^{(\mathrm{A})} - x_f^{(\mathrm{B})}| << 0.4$ m.)

Fig. 2. The trajectories of two curling rocks as predicted by the results in ref. 4. The continuous-line curve shows the path of rock A, the broken-line curve the path of rock B. The y-axis is in the initial direction of motion of rock A. Rock B is released at a time $t_1 = 0.25t_0$ after the release of rock A, where t_0 is the time after its release that rock A stops moving. Rock B is also released such that it moves in the same direction, with the same translational speed, and the same angular speed as rock A; i.e., when rock B is released, the two rocks are moving "side by side," with the same translational and angular velocities. The x'–y' coordinate system shown in the figure has its origin at the release point of rock B, with the y' axis in the initial direction of motion of rock B (see text for full discussion). The approach in ref. 4 predicts that the two rocks will follow different trajectories, and in the case of this figure, will end up with a lateral separation of about 0.4 m! Such predicted motions are in severe disagreement with observed motions of actual curling rocks. See also Fig. 3. Details are given in the text.

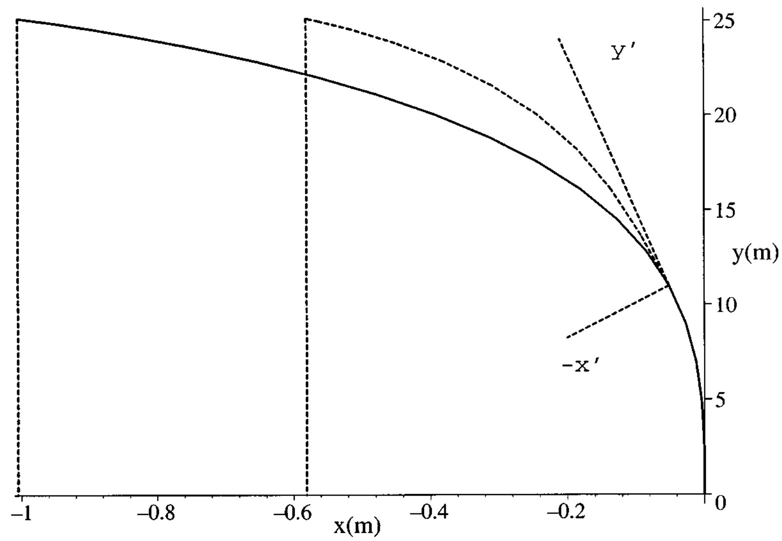
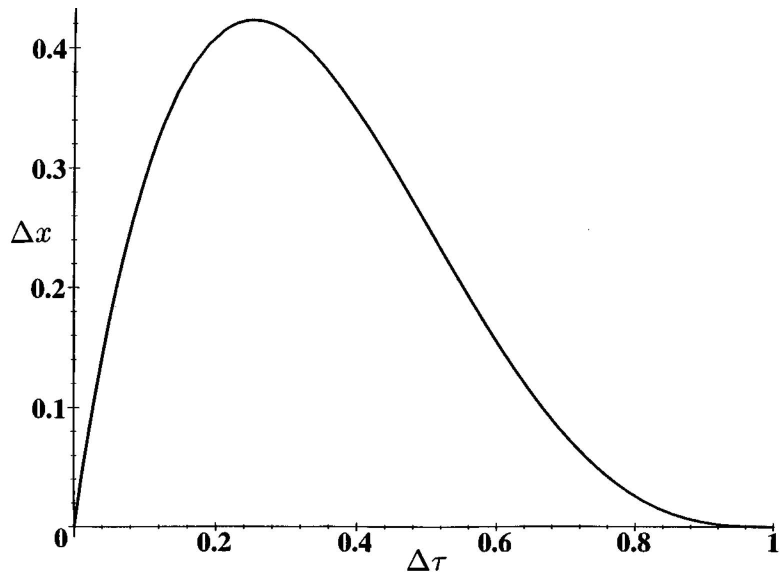


Figure 2 compares the trajectories of rocks A and B, as predicted by the approach and results presented in ref. 4, subsequent to the release of rock B. Note that the trajectories diverge, and the final lateral separation of the rocks is about 0.4 m! This predicted motion of ref. 4 is, clearly, unrealistic.

We emphasize that the trajectories in Fig. 2 begin to diverge *immediately* after the release of rock B. Consequently, the difference in the trajectories *cannot* be attributed to, for example, the breakdown in the approximate equations near the end of the trajectories.

To ensure that there is no misunderstanding about the motions of rocks A and B, we introduce a second inertial frame that has its origin located at the release point of rock B, with its y' axis in the initial direction of motion of rock B, and with its x' axis perpendicular to the initial direction of motion of rock B. This coordinate system is shown explicitly in Fig. 2. Note that the origin of the x'–y' coordinate system, i.e., the point of release of rock B, is at the location of rock A at time t_1 after the release of rock A. According to the results presented in ref. 4, the trajectory of rock B in the x'–y' coordinate system is given by (7) and (8) above. Of course, one must interpret the symbols in these two equations as follows. Equation (7) gives the magnitude of the x'-coordinate of rock B at a time t after the release of rock B. Similarly, (8) gives the y'-coordinate of rock B at a time t after the release of rock B. The symbol v_0 in these equations is the initial speed of rock B, namely, $v_0^{(\mathrm{B})} = v_1 = 1.875$ m/s. The symbol t_0 in these equations is the time after the release of rock B that rock B comes to rest, i.e., $t_0 \equiv t_0^{(\mathrm{B})} \equiv v_0^{(\mathrm{B})}/(\mu g) = v_1/(\mu g)$. Using (7) and (8), one easily determines the final location of rock B in the x'–y' coordinate system: $|x'_{f,(B)}| \equiv |x'_B(t_0^{(\mathrm{B})})| = (br[v_0^{(\mathrm{B})}]^2[t_0^{(\mathrm{B})}]^2)/(24R^2) =$

Fig. 3. The final lateral separation, $\Delta x\,(\mathrm{m})$, of two curling rocks released in the manner of Fig. 2, as a function of the fraction of the elapsed time, $\Delta\tau$, between the release of the two rocks; $\Delta\tau \equiv t_1/t_0$, where t_1 and t_0 are as in Fig. 2. The figure clearly shows that Δx is appreciable for almost all $\Delta\tau$.



$(brv_1^4)/(24R^2\mu^2g^2) \approx 0.318$ m; $y'_{f,(B)} \equiv y'_B(t_0^{(\mathrm{B})}) = \frac{1}{2}v_0^{(\mathrm{B})}t_0^{(\mathrm{B})} = v_1^2/(2\mu g) \approx 14.12$ m. It is a simple matter to convert the final location of rock B from the x'–y' coordinate system into the coordinate system for rock A, i.e., the x–y coordinate system shown in Fig. 2, the origin of which is located at the point of release of rock A, with the y axis in the direction of the initial velocity of rock A, and the x axis perpendicular to the direction of the initial velocity of rock A. For example, $|x_f^{(\mathrm{B})}| \approx |x^{(\mathrm{A})}(t_1)| + |x'_{f,(B)}| + y'_{f,(B)}\psi \approx (0.051 + 0.318 + 14.12 \times 0.0150)\ \mathrm{m} \approx 0.58$ m, where ψ is the magnitude of the angle between the y axis and the y' axis ($\psi \approx 0.0150$).

In Fig. 3 we show the final lateral separation of the two rocks as a function of the time of release, t_1, of rock B after the release of rock A. The figure clearly shows that the final lateral separation is large for all t_1 (except for t_1 close to zero, the time at which rock A is released, or close to the time at which rock A stops moving). So, the large lateral separation is a general feature of the motion predicted by the results of ref. 4, and is not a particular result of the time selected for the release of rock B.

These predicted motions are in disagreement with observed motions of real curling rocks. Moreover, the predicted motions are physically unreasonable; the equations given in ref. 4 imply the following. If one determines the instantaneous velocity and angular velocity of a curling rock at some instant of time *after* the rock has been released, the equations in ref. 4 say that one cannot know what the trajectory of the rock will be *subsequent* to this time; the equations say instead that one needs to know how much time has elapsed since the rock was first released. Clearly, this is physically unreasonable.

To demonstrate the significance of this, we present an alternative way of interpreting the two trajectories in Fig. 2. Instead of regarding the trajectories as belonging to two different rocks, we can interpret them as trajectories predicted by two different observers. Observer “A” sees a rock released with initial speed $v_0^{(\mathrm{A})} = 2.5$ m/s in the direction of the y axis of Fig. 2, released from the origin of the x–y coordinate system in Fig. 2. Using (7) and (8), observer A will predict that the rock will follow the continuous-line trajectory in Fig. 2: this prediction is based on the results presented in ref. 4. At a time t_1 *after* the rock was released from the origin of the x–y coordinate system, a second observer measures the translational speed, rotational speed, and observes the direction of translational motion and direction of rotation of the rock. This observer, observer "B", can then use the results presented in ref. 4 to predict the path of the rock subsequent to observing its "initial" translational and rotational velocities, — i.e., they are initial velocities for observer B. Observer B will then use the x'–y' coordinate system of Fig. 2 to calculate the trajectory of the curling rock, and will predict that it follows the broken-line path of Fig. 2. That the results in ref. 4 imply that observers A and B will predict *different* paths for *the same* curling rock for all times $t > t_1$ clearly reveals that the results of ref. 4 are not self-consistent, and that the work in ref. 4 is wrong.

We emphasize that these physically unreasonable trajectories are a consequence of *only the results presented in ref. 4*; these predicted trajectories have been obtained using *only the equations for* $x(t)$ *and* $y(t)$ *given in ref. 4*.

With regard to the failure of ref. 4, when we refer, for example, to disagreement with observed motions, we are talking about *major qualitative failure*: rocks *do not* follow different trajectories like those in Fig. 2. We are not referring to a small disagreement between prediction and observation, such as, for example, the predicted curl distance being 0.8 that of the observed curl distance. Minor differences like this are to be expected in the course of constructing a good model: predictions of the model allow for experimental and (or) observational tests that in turn allow for improvement in the model. The failure of ref. 4 is not minor.

Later in this paper we will consider the motion of an overturned, rotating, drinking glass sliding over a smooth surface. We will calculate the motion using an inertial frame. We will find that the results are such that one can, as expected, predict the motion subsequent to some instant of time after the glass was released if one knows the velocity and the angular velocity at this time. This is as expected and as required on physical grounds.

That the predicted motions of ref. 4 are in disagreement with observed motions leads to the conclusion that the approach suggested and the results presented in ref. 4 are wrong.

The physical reason for the failure of the approach used in ref. 4 will be discussed in the next section.

5. Left–right asymmetry in a rotating frame

As indicated previously, two reference frames are referred to in ref. 4: an inertial frame, and a rotating, noninertial frame. The origin of the rotating frame was taken to be located directly beneath the center of mass of the rock. The rotating frame was also taken to have an angular speed Ω relative to the inertial frame such that the rock exhibits no lateral motion (as viewed in the rotating frame).

Given the results presented in ref. 4, perhaps the most important question to address is the following: *What is the real, physical force, in the inertial frame, that gives rise to this motion*?

It is straightforward to determine this force, $\boldsymbol{F}$, from (7) and (8) for $x(t)$ and $y(t)$. Calculating the acceleration $\boldsymbol{a}$ in the inertial frame, one finds that

$$\boldsymbol{F} = \boldsymbol{F}_v + \boldsymbol{F}_\mathrm{T} = -\mu M g \boldsymbol{e}_v - M b \frac{r}{R^2}(v_0 - v) v \boldsymbol{e}_\mathrm{T} \tag{9}$$

the transverse component of which has magnitude

$$F_\mathrm{T} = M b \frac{r}{R^2}(v_0 - v) v \tag{10}$$

To ensure clarity, we emphasize that (9) and (10) were obtained *using only the equations for* $x(t)$ *and* $y(t)$ *given in ref. 4*. Consequently, (9) and (10) give the true physical force $\boldsymbol{F}$ required to produce the results presented in ref. 4. As such, the discussion that follows in the remainder of this section will also apply for the next section of this paper.

The same result for $\boldsymbol{F}$ is obtained if one interprets (3) as follows. The relationship between the time rate of change of a vector $\boldsymbol{Q}$, as viewed from two frames, one being the inertial frame and the other the rotating frame, is

$$\left(\frac{\mathrm{d}\boldsymbol{Q}}{\mathrm{d}t}\right)_{\text{fixed}} = \left(\frac{\mathrm{d}\boldsymbol{Q}}{\mathrm{d}t}\right)_{\text{rot}} + \boldsymbol{\Omega} \times \boldsymbol{Q}$$

(See, for example, eq. (10.12), p. 384 of ref. 6, or eq. (10.22), p. 362 of ref. 7, or pp. 84–86 of ref. 8.) If we take $\boldsymbol{Q}$ to be the velocity of the center of mass of the rock relative to the inertial frame, $\boldsymbol{v}$, then we have

$$\boldsymbol{a} \equiv \left(\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t}\right)_{\text{fixed}} = \left(\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t}\right)_{\text{rot}} + \boldsymbol{\Omega} \times \boldsymbol{v}$$

If we next identify $\left(\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t}\right)_{\text{rot}}$ as $\boldsymbol{a}' \equiv \frac{1}{M}\frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\phi\, \boldsymbol{f}(\phi)$, we obtain the same result for $\boldsymbol{a}$, or equivalently, for $\boldsymbol{F}$, as given in (9) above.

The consequence of identifying $\left(\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t}\right)_{\text{rot}}$ in this manner is that the following approach may well have been the approach used in ref. 4: that $\boldsymbol{f}(\phi)$ was considered in ref. 4 to be the real, physical force acting on the contact annulus. But this interpretation would imply that the rock travels in a straight line, which in turn would require $\boldsymbol{\Omega} \equiv \boldsymbol{0}$! (See Sect. 3.) Having $\boldsymbol{\Omega} \neq \boldsymbol{0}$, and given by (5), requires that the true physical force in the inertial frame be given instead by (9).

In other words, if we are to assume that the results given in ref. 4 are correct, then the true, physical force acting on the rock, in the inertial frame, is given by (9) above.

Note that the equation $\boldsymbol{a} = (\mathrm{d}\boldsymbol{v}/\mathrm{d}t)_{\text{rot}} + \boldsymbol{\Omega} \times \boldsymbol{v}$, with $\boldsymbol{F}$ given by (9) and Ω given by (5), is still satisfied, because $(\boldsymbol{F}_{\mathrm{T}})_{\text{rot}} \equiv \boldsymbol{0}$.

Let us be absolutely clear about the meaning of (9) and (10): $\boldsymbol{F}$, in (9), is the force, in the inertial frame, that would be required to give the trajectories presented in ref. 4; $\boldsymbol{F}_{\mathrm{T}}$, given by (10), is the component of the net force exerted on the rock, in an inertial frame, that is transverse to the instantaneous center-of-mass velocity, in the inertial frame.

The following questions must be addressed: *What is the physical origin of the force* $\boldsymbol{F}$? *In particular, what is the physical origin of the transverse component* $\boldsymbol{F}_{\mathrm{T}}$? Nowhere in ref. 4 is there any discussion of the forces in the inertial frame — unless, of course, (1) is to be interpreted as the force in an inertial frame, in which case the rock does not curl. Moreover, nowhere in ref. 4 is there any discussion of the force $\boldsymbol{F}_{\mathrm{T}}$, i.e., the force responsible for the lateral motion of the curling rock!

In problems like the one considered in this paper, namely, the motion of a curling rock, one would usually start by obtaining, in an inertial reference frame, expressions for the forces acting on the object in question. This is what was done in refs. 2 and 3. It is what one would do to obtain a description of, for example, the motion of a rotating, overturned drinking glass sliding over a smooth solid surface (see Sect. 7).

Consequently, the following questions emerge: Does it make sense to start off, in an inertial frame, and write down (9) for the force acting on a curling rock, and (10) for the component transverse to $\boldsymbol{v}$? What is the physical origin of this transverse component of the net force on the rock in the inertial frame? This transverse component of the net force is proportional to the *reduction in speed* of the rock *since it was released*: what is the *physical* reason for such a force?

Again, no physical explanation was given in ref. 4 for the origin of (9) or (10). Instead, a discussion of differential melting around the rock was presented, but that discussion would give a force that is left–right asymmetric *in the inertial frame* (see above).

Related questions that must be addressed are as follows. Why would it be desirable to describe the motion of a curling rock using a rotating, noninertial reference frame? If it was desirable to do so, would it not also be desirable to present the equivalent description in an inertial frame?

We complete this section by summarizing the serious problems with the approach taken and results given in ref. 4.

(1.) The transverse component of the net force on the rock, which has the magnitude

$$F_{\mathrm{T}} = Mb\frac{r}{R^2}(v_0 - v)v \tag{10}$$

is unreasonable from a physical point of view: F_{T} is proportional to $(v_0 - v)$, the amount of speed that has been lost since the rock was released. This means that F_{T} does not depend on only the instantaneous velocity of the center of mass (and possibly also on the instantaneous angular velocity), but is *also* dependent on *the reduction in speed since the rock was released*! This history-dependent transverse force is the principal physical reason that accounts for the different trajectories in Fig. 2, and is the reason why the results in ref. 4 are in severe disagreement with the actual motion of real curling rocks.

(2.) Nowhere in ref. 4 is there any mention of the transverse force $\boldsymbol{F}_{\mathrm{T}}$. The results in ref. 4 cannot be taken seriously unless a derivation of this rather strange force be given. Even with such a derivation, more is needed.

(3.) Additionally, the extra forces around the contact annulus must not only be such that the net force has the transverse piece, $\boldsymbol{F}_{\mathrm{T}}$, but the extra forces must also be taken into account to determine the manner in which the torque equation in ref. 4 must be revised.

(4.) A derivation of (5) for Ω must be given (see below).

(5.) Finally, and most unlikely, observations of motions of actual curling rocks must be made that show that the motion really is history dependent, that rocks actually move as shown in Fig. 2. Such observational evidence seems extremely unlikely, and we conclude that the results in ref. 4 are wrong.

One is thus left to consider the possibility that $\boldsymbol{a}'$ be interpreted instead as the acceleration of the center of mass of the rock in the rotating frame, i.e., that

$$\boldsymbol{a}' \equiv \left(\frac{\mathrm{d}\boldsymbol{v}_{\mathrm{rot}}}{\mathrm{d}t}\right)_{\mathrm{rot}} \equiv \frac{\mathrm{d}\boldsymbol{v}_{\mathrm{rot}}}{\mathrm{d}t}$$

where $\boldsymbol{v}_{\mathrm{rot}}$ is the velocity of the center of mass of the rock in the rotating frame.

This interpretation also fails, as we show in the next section, as it again leads to trajectories that imply motions that are in disagreement with observed motions.

6. Left–right asymmetry in a rotating frame — revisited

Consider next that (1) and (2) are to be interpreted in a second possible way. Specifically, suppose that $\boldsymbol{Q}$ in the equation

$$\left(\frac{\mathrm{d}\boldsymbol{Q}}{\mathrm{d}t}\right)_{\mathrm{fixed}} = \left(\frac{\mathrm{d}\boldsymbol{Q}}{\mathrm{d}t}\right)_{\mathrm{rot}} + \boldsymbol{\Omega} \times \boldsymbol{Q}$$

is taken to be the velocity of the center of mass of the rock relative to the *rotating* frame, $\boldsymbol{v}_{\mathrm{rot}}$, then we have

$$\left(\frac{\mathrm{d}\boldsymbol{v}_{\mathrm{rot}}}{\mathrm{d}t}\right)_{\mathrm{fixed}} = \left(\frac{\mathrm{d}\boldsymbol{v}_{\mathrm{rot}}}{\mathrm{d}t}\right)_{\mathrm{rot}} + \boldsymbol{\Omega} \times \boldsymbol{v}_{\mathrm{rot}} \equiv \boldsymbol{a}_{\mathrm{rot}} + \boldsymbol{\Omega} \times \boldsymbol{v}_{\mathrm{rot}}$$

The velocities $\boldsymbol{v}$ and $\boldsymbol{v}_{\mathrm{rot}}$ are related by the following equation:

$$\boldsymbol{v} = \boldsymbol{v}_{\mathrm{rot}} + \boldsymbol{\Omega} \times \boldsymbol{r}_{\mathrm{rot}}$$

where $\boldsymbol{r}_{\text{rot}}$ is the location of the center of mass in the rotating frame. (This equation is given in most any upper year undergraduate textbook on classical mechanics; for example, eq. (10.17), p. 385 of ref. 6, or eq. (10.23), p. 362 of ref. 7.) Combining these last two equations gives

$$\boldsymbol{a} = \boldsymbol{a}_{\text{rot}} + 2\boldsymbol{\Omega} \times \boldsymbol{v}_{\text{rot}}$$

(note that, while $\boldsymbol{r}_{\text{rot}} = \boldsymbol{0}$, $\dot{\boldsymbol{r}}_{\text{rot}} \neq \boldsymbol{0}$).

With this interpretation, (1) and (2) give the *effective* force, $\boldsymbol{F}_{\text{eff}}$, acting on the rock as seen in the rotating frame; i.e., multiplying the equation for $\boldsymbol{a}$ above by M gives

$$\boldsymbol{F} = \boldsymbol{F}_{\text{eff}} + 2M\ \boldsymbol{\Omega} \times \boldsymbol{v}_{\text{rot}}$$

where

$$\boldsymbol{F}_{\text{eff}} = M\boldsymbol{a}_{\text{rot}} = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\phi\, \boldsymbol{f}(\phi)$$

The crucial question, again, is: *What is the true force acting on the rock, in the inertial frame*? The answer is, except for the factor of 2, the same as in the previous section. The resultant motion is, again, physically unacceptable. The five serious problems listed at the end of Sect. 5, therefore, apply for both possible interpretations of $\boldsymbol{f}(\phi)$.

It would seem that the principal problem with ref. 4 is that $\boldsymbol{\Omega}$ has not been properly calculated (see below). Indeed, it seems odd that one would want to use rotating frames to do the calculation at all. To demonstrate this point, we will consider a simpler motion, one where the forces involved are unequivocal, and we show that the motion is easily analyzed in an inertial frame, and that there is little to gain by working in a rotating frame. The motion we will study (in the next section) is that of an overturned, rotating drinking glass that slides over a smooth surface.

The conclusion is the following. Whether one interprets the force used in the approach taken in ref. 4 as the real physical force in an inertial frame, or the effective force in a rotating noninertial frame, the consequence is the same: *the trajectories that result, predict motions of curling rocks that are in disagreement with the observed motions of actual curling rocks.*

The only reasonable conclusion, therefore, is that the approach suggested in ref. 4, and the results presented in ref. 4, are wrong.

7. Motion of a sliding, rotating cylinder: inertial vs. rotating frames

We have shown that the motion in ref. 4 is physically unreasonable, and that the results given in ref. 4 are incorrect.

The following questions remain to be addressed. Can one work exclusively in a rotating frame, and correctly solve for all aspects of the motion? Is there anything to be gained by working in a rotating frame instead of an inertial frame? In posing these questions, we are referring to the motions of rotating cylinders sliding over solid surfaces.

The reason we address these questions is as follows. We have seen that the force $\boldsymbol{f}(\phi)$ given by (1), and used in ref. 4, can be interpreted in two ways: (*i*) $\boldsymbol{f}(\phi)$ is the true, physical force in the inertial frame; (*ii*) $\boldsymbol{f}(\phi)$ is the effective force in the rotating, noninertial frame. It is difficult to believe that possibility (*i*) was employed in ref. 4 because, as has been unequivocally demonstrated earlier in this paper, this interpretation results in straight line motion of the curling rock, which is not what was reported in ref. 4. We must therefore conclude that possibility (*ii*) was what was actually tried in ref. 4. Indeed, in using the equation $\boldsymbol{a} = \boldsymbol{a}' + \boldsymbol{\Omega} \times \boldsymbol{v}$ (i.e., (3)), it seems that the strategy in ref. 4 was to calculate the terms on the right-hand side to find $\boldsymbol{a}$, and thereby also find $x(t)$ and $y(t)$. In other words, the true physical force $\boldsymbol{F}$ was not used in ref. 4 to find $\boldsymbol{a}$. Instead, an attempt was made to calculate $\boldsymbol{a}'$, $\boldsymbol{\Omega}$, and $\boldsymbol{v}$, apparently by working in a rotating frame. Consequently, we address the questions stated above.

Our principal objective in this section is to use a simple example of a sliding, rotating cylinder to illustrate the difficulties inherent in working *exclusively* in a rotating reference frame. In doing so, we provide an instructive example that clearly illustrates that it is straightforward to solve problems of this nature by working exclusively in an inertial frame.

We will first calculate the motion in an inertial frame. Then we will inquire as to how the calculation might be attempted in a rotating frame. We will see that nontrivial problems can arise in trying to calculate the effective external forces on the cylinder in the rotating frame. Our conclusion will be that it is best to work in an inertial frame, and to use the true physical force to solve the problem.

We consider the case of a cylinder rotating about its center of mass and sliding over a smooth solid surface. The surface is such that no melting is involved. An example is the motion of a cylindrically symmetric overturned drinking glass. The equations of motion for such a case are unequivocal. One simply considers the friction and normal force acting around the contact annulus and derives equations for the net external force and the torque acting on the glass; one then obtains expressions for the acceleration of the center of mass and the angular speed of rotation. All of this is readily done in an inertial frame.

We present here the results of such a calculation. To simplify the discussion, we consider the case of slow rotation ($r\omega_0 << v_0$) and small lateral displacement as compared to the distance traveled. The y-axis of the inertial frame is in the direction of the initial velocity $\boldsymbol{v}_0$ of the center of mass of the glass, the x-axis is perpendicular to the y-axis, and the z-axis is normal to and away from the surface. We define unit vectors $\boldsymbol{e}_v$ and $\boldsymbol{e}_\mathrm{T}$ in the inertial frame as follows: $\boldsymbol{e}_v$ is in the direction of the instantaneous velocity of the center of mass, $\boldsymbol{e}_\mathrm{T}$ is transverse to $\boldsymbol{e}_v$ and such that $\boldsymbol{e}_\mathrm{T} \times \boldsymbol{e}_v = \boldsymbol{e}_z$, where $\boldsymbol{e}_z$ is a unit vector in the $+z$-direction.

The normal force acting around the contact annulus of the overturned glass is given by

$$\mathrm{d}\boldsymbol{N}(\theta) = \mathrm{d}N(\theta)\boldsymbol{e}_z = \frac{1}{2\pi}\left(1 + \frac{2h\mu}{r}\sin\theta\right)Mg\mathrm{d}\theta\boldsymbol{e}_z \tag{11a}$$

where h is the height of the center of mass of the glass above the surface, μ is the coefficient of kinetic friction, r is the radius of the contact annulus of the glass, and the angle θ is measured counterclockwise from $\boldsymbol{e}_\mathrm{T}$. The friction exerted by the surface on a small portion of the glass, at angle θ, has magnitude $\mu\mathrm{d}N(\theta)$ and is in the direction opposite to the velocity of the small portion relative to the surface; for the case of clockwise rotation (as viewed from above), to leading order, the components are given by

$$\mathrm{d}\boldsymbol{F}_\mathrm{T}(\theta) \approx -\mu\frac{r\omega}{v}\sin\theta\mathrm{d}N(\theta)\boldsymbol{e}_\mathrm{T} \tag{11b}$$

$$\mathrm{d}\boldsymbol{F}_v(\theta) \approx -\mu\mathrm{d}N(\theta)\boldsymbol{e}_v \tag{11c}$$

the contribution to the torque is

$$\mathrm{d}\boldsymbol{\tau}_z(\theta) \approx -\mu r\cos\left(\theta + \frac{r\omega}{v}\sin\theta\right)\mathrm{d}N(\theta)\boldsymbol{e}_z \tag{11d}$$

One readily verifies using equations (11) that the net force on the glass in the z-direction is zero, and that the net torque on the glass in the plane of the surface is also zero, as required. For the case of slow rotation, one also readily finds the equations for the components of the net force along and transverse to the direction of the instantaneous velocity of the center of mass, as well as the net torque. For clockwise rotation of the glass, as viewed from above, one finds:

$$\boldsymbol{F}_v \approx -\mu Mg\boldsymbol{e}_v \tag{12}$$

$$\boldsymbol{F}_\mathrm{T} \approx -\mu^2 Mgh\frac{\omega(t)}{v(t)}\boldsymbol{e}_\mathrm{T} \tag{13}$$

and

$$\boldsymbol{\tau}_z \approx \frac{1}{2}\mu M g r^2 \frac{\omega(t)}{v(t)} \boldsymbol{e}_z \tag{14}$$

where $\omega(t)$ and $v(t)$ are the magnitudes of the angular speed and the center of mass speed at time t.

Using $\boldsymbol{F} = M\mathrm{d}\boldsymbol{v}/\mathrm{d}t$ and $\boldsymbol{\tau}_z = I\mathrm{d}\boldsymbol{\omega}/\mathrm{d}t$ and taking $I = \frac{1}{2}Mr^2$, we find for the magnitudes of the center-of-mass speed v and the angular speed ω:

$$v(t) = v_0\left(1 - \frac{t}{t_0}\right), \qquad t_0 \equiv \frac{v_0}{\mu g} \tag{15}$$

$$\omega(t) = \omega_0\left(1 - \frac{t}{t_0}\right) \tag{16}$$

These results are to leading order only. Also to leading order, we find the distance traveled in the y-direction is given by

$$y(t) = v_0 t\left(1 - \frac{1}{2}\frac{t}{t_0}\right) \tag{17}$$

The x-component of the force is given by

$$F_x = -F_{\mathrm{T}} \cos\psi_v + F_v \sin\psi_v \approx -\frac{\mu^2 M g h \omega(t)}{v(t)} + \mu M g \psi_v(t) \tag{18}$$

where $\psi_v(t)$ is the magnitude of the angle made by the y-axis and the center-of-mass velocity $\boldsymbol{v}$, and F_{T} and F_v are the magnitudes of $\boldsymbol{F}_{\mathrm{T}}$ and $\boldsymbol{F}_v$. For the case of slow rotation, we have

$$\psi_v(t) \approx \frac{|v_x(t)|}{v_y(t)}$$

Thus, to leading order, the equation for $x(t)$ is

$$\frac{\mathrm{d}^2 x(t)}{\mathrm{d}t^2} = -\mu g\left[\mu h\frac{\omega_0}{v_0} + \frac{1}{v_0 - \mu g t}\frac{\mathrm{d}x(t)}{\mathrm{d}t}\right] \tag{19}$$

The solution to (19) is

$$x(t) = -\frac{h\omega_0 v_0}{g}\frac{1}{2}\left[(1-\tau)^2\left(\ln(1-\tau) - \frac{1}{2}\right) + \frac{1}{2}\right] \tag{20}$$

where $\tau \equiv t/t_0$. One readily verifies that this expression satisfies (19).

It is also straightforward to show that these equations for $x(t)$ and $y(t)$ pass the "trajectory test" used earlier to show that the results in ref. 4 are physically unacceptable. Here, one finds that the trajectories of the two drinking glasses are identical. Consequently, as expected and as required, one can determine the motion subsequent to any time t (with $0 < t < t_0$) given that one knows $\boldsymbol{v}(t)$ and $\boldsymbol{\omega}(t)$. This follows from the form of equations (12)–(14).

We complete this calculation by giving typical values for the motion considered. For example, taking $\mu = 0.05$, $h = 0.12$ m, $\omega_0 = 2\mathrm{s}^{-1}$, $v_0 = 0.5$ m/s, and using $g = 9.8$ m/s^2, we have $y(t_0) \approx 25.5$ cm and $|x(t_0)| \approx 0.3$ cm.

All of the above has been done exclusively in the inertial frame.

We next address the question of what should be done in a rotating frame.

One might try to obtain an expression for the *effective force* of friction exerted on the contact annulus *in the rotating frame*, i.e., to derive an expression for the effective force $\mathrm{d}\boldsymbol{f}_{\mathrm{eff}}(\theta)$ around the contact annulus. One might attempt this by using the following equation (see, for example, eq. (10.23), p. 386 of ref. 6, or eq. (10.26), p. 363 of ref. 7), which relates the acceleration $\boldsymbol{a}$ in the inertial frame and the acceleration $\boldsymbol{a}_{\mathrm{rot}}$ in the rotating frame:

$$\boldsymbol{a} = \boldsymbol{a}_{\mathrm{rot}} + \dot{\boldsymbol{\Omega}} \times \boldsymbol{r}_{\mathrm{rot}} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{r}_{\mathrm{rot}}) + 2\boldsymbol{\Omega} \times \boldsymbol{v}_{\mathrm{rot}} \tag{21}$$

where the location $\boldsymbol{r}_{\mathrm{rot}}$ and the velocity $\boldsymbol{v}_{\mathrm{rot}}$ of the point in question are measured in the rotating frame, $\boldsymbol{a}_{\mathrm{rot}} \equiv (\mathrm{d}\boldsymbol{v}_{\mathrm{rot}}/\mathrm{d}t)_{\mathrm{rot}}$, $\boldsymbol{v}_{\mathrm{rot}} \equiv (\mathrm{d}\boldsymbol{r}_{\mathrm{rot}}/\mathrm{d}t)_{\mathrm{rot}}$, and $\dot{\boldsymbol{\Omega}} \equiv \mathrm{d}\boldsymbol{\Omega}/\mathrm{d}t$.

The result of this is that one obtains expressions for the net effective force and the net torque, in the rotating frame, and one finds that the equation for the torque is not correct.

The physical reason for the failure of this approach is as follows. Equation (21) relates the *accelerations* in the two reference frames, *not the external forces*. For example, if one considers a portion of the contact annulus, at angle θ, with mass $\mathrm{d}m(\theta)$, one might think it is related to $\mathrm{d}\boldsymbol{f}(\theta)$ by $\mathrm{d}\boldsymbol{f}(\theta) = \mathrm{d}m(\theta)\boldsymbol{a}(\theta)$, where $\mathrm{d}\boldsymbol{f}(\theta)$ is the net force of friction on the mass element $\mathrm{d}m(\theta)$. This is not correct; the correct expression is, or course, $\mathrm{d}\boldsymbol{f}_{ext}^{\mathrm{NET}}(\theta) = \mathrm{d}m(\theta)\boldsymbol{a}(\theta)$, where $\mathrm{d}\boldsymbol{f}_{ext}^{\mathrm{NET}}(\theta)$ is the net, external force on the mass element, and is not just due to the external force of friction but also includes internal forces.

Correct expressions for the net force and the net torque may be obtained from the accelerations, provided proper care is taken. Note, however, that one must use knowledge of the motion in the inertial frame in order to obtain correct results in the rotating frame.

Another approach to doing the calculation in the rotating frame is to guess, or hypothesize, what the external, effective force of friction in the rotating frame might be. For example, one could try to take $\mathrm{d}\boldsymbol{f}_{\mathrm{eff}}(\theta)$ to be the same function of $\boldsymbol{v}_{\mathrm{rot}}$, $\boldsymbol{\omega}_{\mathrm{rot}}$, θ', etc., as $\mathrm{d}\boldsymbol{f}(\theta)$ is of $\boldsymbol{v}$, $\boldsymbol{\omega}$, θ, etc. This also fails to give a complete and correct description of the motion. Alternatively, one could try to take $\mathrm{d}\boldsymbol{f}_{\mathrm{eff}}(\theta)$ to be similar, but not identical to $\mathrm{d}\boldsymbol{f}(\theta)$. The "obvious" choices (which we have looked at) also fail.

We have examined these and other similar problems in detail. The interested reader will no doubt benefit from examining such problems.

One could try to adopt the approach used in ref. 4. There, instead of using the equations $\boldsymbol{F} = M\mathrm{d}\boldsymbol{v}/\mathrm{d}t$ and $\boldsymbol{\tau} = I\mathrm{d}\boldsymbol{\omega}/\mathrm{d}t$ (which are three equations in the three unknowns $[v_x(t)$, $v_y(t)$, and $\omega(t)]$), to solve for the motion, an attempt was made to obtain the acceleration $\boldsymbol{a}$ in the inertial frame by solving for the quantities on the right-hand side of the equation $\boldsymbol{a} = \boldsymbol{a}' + \boldsymbol{\Omega} \times \boldsymbol{v}$. We have seen earlier in this paper how expressions for $\boldsymbol{a}'$ and $\boldsymbol{v}$ were obtained in ref. 4. The following approach was taken in ref. 4 to try to determine Ω.

The equation $\tau = I\mathrm{d}\omega/\mathrm{d}t = I(\mathrm{d}\omega_{\mathrm{rot}}/\mathrm{d}t + \mathrm{d}\Omega/\mathrm{d}t)$ with $\boldsymbol{\tau} = \frac{1}{2\pi}\int_0^{2\pi} \boldsymbol{r}(\phi) \times \boldsymbol{f}(\phi)\mathrm{d}\phi$, was used in ref. 4 to obtain expressions for both ω_{rot} *and* Ω. The term not involving b was used to give ω_{rot}, while the term proportional to b was used to give an expression for Ω.

If one attempts to use the analogous approach for the overturned drinking glass, one obtains an expression for Ω that is completely incorrect. Specifically, identifying the leading term in τ that involves the asymmetry in the force around the contact annulus to give Ω, as was done in ref. 4, gives an incorrect result for Ω.

One can try many different approaches to extract the equations of motion by working exclusively in the rotating frame. However, one always has the number of unknowns being one more than the number of equations. In other words, if one does not know the motion in the inertial frame, one cannot calculate *all* aspects of the motion in the rotating frame. It seems to us that one needs some knowledge of the motion in the inertial frame to get complete and correct results using the rotating frame.

To ensure clarity, we point out that one can, of course, benefit by using a rotating frame in combination with an inertial frame and the true physical force. For example, consider again (18) for F_x. Using a rotating frame, the origin of which is directly beneath the instantaneous position of the center of mass, and the equation

$$\boldsymbol{a}_{\mathrm{T}} = \boldsymbol{\Omega} \times \boldsymbol{v}$$

one finds from (13), (15), and (16) that

$$\Omega(t) = \frac{\mu^2 g h \omega_0}{v_0} \frac{1}{v(t)} = \frac{\mu^2 g h \omega_0}{v_0} \frac{1}{v_0 - \mu g t}$$

One then finds that

$$\psi_v(t) = \int_0^t \Omega(t')\mathrm{d}t' = -\frac{\mu h \omega_0}{v_0} \ln\left(1 - \frac{\mu g}{v_0} t\right)$$

Using this in (18) gives

$$\frac{\mathrm{d}^2 x(t)}{\mathrm{d}t^2} = -\frac{\mu^2 g h \omega_0}{v_0}\left[1 + \ln\left(1 - \frac{\mu g}{v_0} t\right)\right] \tag{22}$$

One readily solves (22) to obtain $x(t)$ as given by (20). One could regard this manner of solving for $x(t)$ as easier than solving (19). As such, there is arguably some benefit to using a rotating frame. However, we emphasize that this method requires knowledge of the true physical force in the inertial frame. We also point out that, again, this is not what was done in ref. 4. For completeness, we note that these approximate equations break down as $t \to t_0$.

Based on the results for the simple case of an overturned drinking glass sliding over a smooth solid surface, we conclude the following. Since problems can arise, even in this simple case, in trying to obtain an expression for the effective force around the contact annulus *in the rotating frame*, it does not seem to be a good idea to try to describe the considerably more complicated problem of the motion of a curling rock using a rotating reference frame. One cannot use (21) to obtain the effective force due to friction on the contact annulus in the rotating frame. Taking the effective force to have *the same* expression as the inertial force, as was apparently done in ref. 4, also seems not to be a physically meaningful approach.

We conclude that knowledge of the true, physical forces is required in order to completely and correctly calculate all aspects of the motion. It would therefore seem best to address problems of this type by working in an inertial frame, where the forces on the object are known, or at least wherein a physically sound approach may be used.

8. Discussion

Our two principal conclusions are that the approach taken and results presented in ref. 4 are incorrect, and that it is much better to work in an inertial frame than to address the motion of a curling rock by working in a noninertial frame. We expound upon these conclusions in the next section.

In this section, we focus on various other aspects of ref. 4 which need to be commented on.

(1) It is claimed in ref. 4 that a "simple" left–right asymmetry accounts for the motion of curling rocks; upon realizing that the results in ref. 4 are incorrect, we see that the issue of simplicity is irrelevant.

(2) The following statement is made in ref. 4: "Linear and angular velocity cease at the same time, irrespective of their initial values." This statement is not in accord with observations. Rapidly rotating curling rocks (i.e., $r\omega_0 >> v_0$) cease translational motion long before rotational motion stops [9]. This behaviour is, indeed, a prediction of the model in ref. 2, and has been verified by direct observation of actual curling rocks [9].

(3) Regarding the same statement, namely that translational and rotational motions stop at the same time, it is claimed in ref. 4 that this simultaneous cessation is *proved* in ref. 4. This is not so. In ref. 4, only approximate expressions are given. However, we emphasize that *approximate* expressions break down prior to cessation of rotational and translational motions. [2] To *prove* that the two motions stop at the same instant, one must present *exact* results, not approximations. That approximate solutions give simultaneous cessation of rotational and translation motion is certainly not proof that exact solutions will give the same result. An exact proof has been given for a hockey puck on flat ice [10]. Exact results for curling rocks are given in ref. 9.

(4) It is also claimed in ref. 4 that the total curl distance is insensitive to the initial angular speed. Let us be absolutely clear: this is *an opinion*, it is *not* an established scientific fact. Moreover, this opinion is *not* shared by *all* competent, capable curlers. In any case, before it is to be taken as *fact*, experimental data need to be taken. (This is, indeed, recognized in ref. 4.) Experimental data are required to establish how the curl distance depends on the initial angular speed. Even so, it is claimed in ref. 4 that the proposal made there is "successful" in giving a total curl distance that does not depend on the initial angular speed, ω_0, of the rock. Since (1) and (2) have no ω_0 dependence, the result is inevitable. Even if a derivation of the transverse force $\boldsymbol{F}_{\mathrm{T}}$ of the net force on the rock in the inertial frame [i.e., (10)] is eventually physically justified, or, more likely, if some similar but correct alternative proposal is made, the following must also be addressed. It is argued in ref. 4 that the left–right asymmetry is due to one side of the rock moving faster over the ice than the other: this is the origin of the parameter b. Since it is the rotational speed, $r\omega$, that gives rise to the asymmetry in (2), it seems that the coefficient b would have ω dependence. However, in ref. 4, b was taken to be a constant; no physical justification for this was given. *Why* is b independent of ω? Since we require $b \to 0$ as $\omega \to 0$, why would not b increase with ω for small ω? What is the *physical* reason? Why is b insensitive to the value of ω_0; i.e., what is the *physical* reason?

(5) Continuing with the question of the ω_0 dependence of the total curl distance, some who have read ref. 4 might think that a good test to compare refs. 2 and 4 might be to experimentally determine how the curl distance depends on ω_0. If indeed the analysis in ref. 4 is incorrect, there is no point in such a comparison. Also, as noted above, the insensitivity of the curl distance on ω_0 in ref. 4 is not a *consequence* of the approach in ref. 4: it is inevitable, since it was *assumed* in the approach adopted in ref. 4. Experimental results indicating how the curl distance depends on ω_0 are, of course, desirable.

(6) It is also claimed in ref. 4 that the models given in refs. 2 and 3 lack predictive power. This claim is also false. The model presented in ref. 2 predicts that rapidly rotating curling rocks will cease translational motion well before rotation stops. Observations confirm this prediction (details are given in ref. 9). Moreover, the model of ref. 2 further predicts that cylinders having a contact geometry that is very different than that of curling rocks will reveal just the opposite behaviour: the rotational motion stops well before the translational motion [11]. The contact geometry in this case is a number of contact segments that are evenly spaced around the outside of the cylinder and are all oriented radially outward from the center of the cylinder. This prediction has also recently been verified. (Details are given in ref. 11.) Both predictions are quite significant, and strongly support the model presented in ref. 2.

(7) Finally, it is not clear that Ω was actually *solved for* in ref. 4. It appears that only two equations were

[2] Such approximations assume that the rotational speed $r\omega$ is much smaller than the center of mass speed v. This is true for slowly rotating curling rocks ($r\omega_0 << v_0$) for most of the duration of their motion. The approximation eventually fails when $r\omega$ becomes of order v. For further discussion, see ref. 9.

given for the three unknowns: $v(t)$, $\omega_{\rm rot}(t)$, and $\Omega(t)$. Specifically, the two equations that were used in ref. 4 are: $-\mu g = \dot{v}$ and $\boldsymbol{\tau} = I(\dot{\boldsymbol{\omega}}_{\rm rot} + \dot{\boldsymbol{\Omega}})$, where $\boldsymbol{\tau}$ is given by $\boldsymbol{\tau} = \frac{1}{2\pi}\int_0^{2\pi} \boldsymbol{r}(\phi) \times \boldsymbol{f}(\phi)\mathrm{d}\phi$ and $\boldsymbol{\omega}_{\rm rot}$ is the angular acceleration of the curling rock in the rotating frame. The former was used to determine the center-of-mass speed (i.e., $v = v_0 - \mu g t$). It would seem that the latter equation would be used to calculate $\omega(t)$ (i.e., $\boldsymbol{\tau} = I\dot{\boldsymbol{\omega}}$); however, the torque equation was used in ref. 4 to extract results *for both* $\omega_{\rm rot}(t)$ *and* $\Omega(t)$. Clearly, three equations are needed to solve for three unknowns; i.e., a third equation is needed to *calculate* $\Omega(t)$. As shown in sect. 7, it seems that nontrivial problems arise when one attempts to calculate all three quantities if one works exclusively in the rotating frame. However, all three [i.e., $v(t)$, $\omega(t)$, and $\Omega(t)$] are readily calculated by working in an inertial frame. The argument presented in ref. 4 that appears to give (5) for Ω seems to be the following: since the rock goes in a straight line for $b = 0$ (i.e., the "rotating" frame and the inertial frame coincide at all times), the term involving b when $b \neq 0$ in the equation $\boldsymbol{\tau} = I(\dot{\boldsymbol{\omega}}_{\rm rot} + \dot{\boldsymbol{\Omega}})$ must give Ω. This is not a convincing argument. One could equally well argue as follows: whether $b = 0$ or $b \neq 0$, since the true, physical force is left–right asymmetric, the rock goes in a straight line, and the "rotating" frame and the inertial frame coincide at all times, i.e., $\Omega \equiv 0$. If any proposal like the one made in ref. 4 is to taken as a reasonable description of the motion of a curling rock, it is necessary that $\Omega(t)$ *be calculated*, and in such a manner as to leave no doubt about the result. It is *also* required that a physical derivation be given of the associated lateral force $\boldsymbol{F}_{\rm T}$ in the inertial frame, because $\Omega(t)$ and $\boldsymbol{F}_{\rm T}$ are inextricably linked. Moreover, the force $\boldsymbol{F}_{\rm T}$ must be *physically reasonable*.

9. Conclusions

Our two main conclusions in this paper are

[1] The approach taken and the results presented in ref. 4 are incorrect.

[2] For the purpose of studying the motion of curling rocks, it is much better to work in an inertial frame than to attempt to solve the problem using a noninertial frame.

We next expound upon these results. The principal results of this paper are as follows:

(1) We have shown that, if one interprets the left–right asymmetric force given in ref. 4 to be in the inertial frame, then the curling rock would move in a straight line.

(2) If instead we interpret the left–right asymmetric force to be in the rotating frame, the lateral motion of a curling rock does not result from left–right asymmetry in the inertial frame, and instead is due to the transverse component $\boldsymbol{F}_{\rm T}$, given by (10), of a real force in the inertial frame. In this interpretation, no physical basis was given in ref. 4 for the description proposed in ref. 4. Consequently, the proposal made in ref. 4 cannot be taken as a legitimate model of the motion of a curling rock unless several conditions are met (see Sect. 5); it is extremely unlikely that all these conditions can be met. It is however possible that a similar but alternative approach may result in reasonable trajectories. Such an approach would have to meet the conditions specified in Sect. 5. One of these conditions, for example, is that *a physical derivation* of $\boldsymbol{F}_{\rm T}$, in an inertial frame, must be presented: if the force on a curling rock actually has left–right asymmetry *in the rotating frame*, then it must be derived by *also* deriving the forces that act on the rock *in an inertial frame*. The other requirements are given in Sect. 5. The most important requirement is of course that the trajectories be *physically reasonable* in the sense described earlier in this paper.

(3) Based on our results for the straightforward case of a rotating cylinder sliding over a surface for which there is no melting, we conclude that the considerably more complicated problem of the motion of a curling rock is best analyzed in an inertial frame, and not in a rotating, noninertial frame.

(4) Perhaps *the most important result of this paper*, is that we have shown that, the approach taken and the results presented in ref. 4 lead to *predictions of motions of curling rocks* that are *in disagreement with motions of real curling rocks*.

We have carefully read the Reply to Comment following our paper. In our opinion, nothing of merit is stated in the Reply. Instead of addressing all of the issues raised there, we simply invite the readers to carefully read our Comment and the Reply to Comment, and carefully consider both points of view. We leave it to the readers to decide for themselves what is correct and what is incorrect

An important consequence of the work we have reported in this paper, is that models such as refs. 2 and 3 remain as viable candidates for the explanation of the motion of curling rocks. We note that differential melting most probably does occur around the contact annulus; but differential melting *alone* simply cannot account for the curl of a curling rock. In our view, it is the tendency of the granite–liquid adhesion to draw some of the thin liquid film around the rock [2] that accounts for most of the curl, even if differential melting does occur. The dragging of the liquid to the front of the rock thus gives front–back asymmetry, resulting in lateral motion. As noted in this paper, various nontrivial predictions of this model have been made and confirmed by actual observation and (or) experimentation. Details are given elsewhere (see, for example, refs. 2, 9, and 11).

At this time, it seems that front–back asymmetry is certain to play a major role in what will be the ultimate description of curling rock motion.

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